113 Class Problems: Characteristic and Ring Extensions

1. (a) Give and example of a characteristic zero field which is **not** $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

   (b) Give an example of a ring R and two element $a, b \in R \setminus \{1_R\}$ such that $a \neq b$ and $ord(a) \neq ord(b)$.

   (c) If R is an integral domain and $I \subset R$ is an ideal such that R/I is an integral domain, is it necessarily true that $Char(R) = Char(R/I)$?

   (d) If F is a finite field and $Char(F) = p$, prove that $|F| = p^n$

   Solutions:

a) $\mathbb{Q}(x) := Frac(\mathbb{Q}[x])$

b) $[2], [3] \in \mathbb{Z}/4\mathbb{Z}$

Prime

c) No $\quad char(\mathbb{Z}) = 0$. $\quad Char(\mathbb{Z}/p\mathbb{Z}) = p$

d) $Char(F) = p \Rightarrow ord(a) = p \quad \forall\, a \neq 0_R$

isomorphism of groups by Structure Theorem

$\Rightarrow (F, +) \cong \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$

$\Rightarrow |F| = p^n$

2. Determine all units in $\mathbb{Z}[i]$.

Solutions:

$\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\} \Rightarrow |\alpha| \geq 1 \quad \forall \alpha \in \mathbb{Z}[i]$

$\alpha \in \mathbb{Z}[i]^* \Leftrightarrow \exists\, \beta \in \mathbb{Z}[i]$ such that $\alpha\beta = 1$

$\Rightarrow |\alpha| \cdot |\beta| = 1 \Rightarrow |\alpha| = 1 \Rightarrow \alpha = 1, -1, i, -i$

$\Rightarrow \mathbb{Z}[i] = \{1, -1, i, -i\}$

3. (a) Prove that $\mathbb{Q}[i] = \mathbb{Q}(i)$

   (b) Prove that $\mathbb{Q}[\sqrt{2}+\sqrt{3}] = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$

Solutions:

$a,b,c,d \in \mathbb{Q}$

a) $\dfrac{a+bi}{c+di} = (a+bi)\left(\dfrac{c}{c^2+d^2} - \dfrac{d}{c^2+d^2}\right) i \in \mathbb{Q}[i]$

$\Rightarrow \mathbb{Q}(i) \subset \mathbb{Q}[i]$.

$\mathbb{Q}[i] \subset \mathbb{Q}(i)$ trivially $\Rightarrow \mathbb{Q}(i) = \mathbb{Q}[i]$

b) $\mathbb{Q}[\sqrt{2}+\sqrt{3}] \subset \mathbb{Q}[\sqrt{2}, \sqrt{3}]$ trivially

$(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6} \Rightarrow (\sqrt{2}+\sqrt{3})^3 = 11\sqrt{2} + 9\sqrt{3}$

$$\Rightarrow \sqrt{2} = \frac{(\sqrt{2}+\sqrt{3})^3 - 9(\sqrt{2}+\sqrt{3})}{2} \in \mathbb{Q}[\sqrt{2}+\sqrt{3}]$$

$$\sqrt{3} = \frac{(\sqrt{2}+\sqrt{3})^3 - 11(\sqrt{2}+\sqrt{3})}{-2} \in \mathbb{Q}[\sqrt{2}+\sqrt{3}]$$

$\Rightarrow \mathbb{Q}[\sqrt{2}, \sqrt{3}] \subset \mathbb{Q}[\sqrt{2}+\sqrt{3}]$